STAT0041: Stochastic Calculus

Lecture 10 - Properties of Itô Integral

Lecturer: Weichen Zhao | Fall 2025

Key concepts:

- *Itô*等距；
- *Itô*积分的鞅性。

10.1 Itô积分的基本性质

Proposition 10.1 设$\mathscr{L}_T^2$是$(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathrm{P})$上平方可积的$\mathscr{F}_t$适应过程的集合，对于$H, G \in \mathscr{L}_T^2$以及 $a, b \in \mathbb{R}$，有

(1) (Mean zero) $\mathbb{E}\left[\int_0^T H_t \mathrm{d}B_t\right] = 0$;

(2) (Itô Isometry) $\mathbb{E}\left[\left(\int_0^T H_t \mathrm{d}B_t\right)^2\right] = \mathbb{E}\left[\int_0^T |H_t|^2 \mathrm{d}t\right]$;

(3) (Linearity) $\int_0^T (aH_t + bG_t)\,\mathrm{d}B_t = a\int_0^T H_t \mathrm{d}B_t + b\int_0^T G_t \mathrm{d}B_t$.

Proof: 我们已经证明过这些性质对简单阶梯过程都是成立的。那么只需要通过简单过程序列逼近H，取极限即可证明。 ■

10.2 Itô积分作为随机过程

设 $H \in \mathscr{L}_T^2$，对于$t \in [0, T]$，定义一个随机过程

$$\mathcal{I}_t[H] \triangleq \int_0^t H_s \mathrm{d}B_s = \int_0^T H_s \mathbf{1}_{[0,t]}(s) \mathrm{d}B_s.$$

我们称随机过程$(\mathcal{I}_t[H])_{t\in[0,T]}$是适应过程$H_t$关于布朗运动$B_t$的不定积分。

下面定理说明Itô积分是一个鞅。

Theorem 10.2 令 $H \in \mathscr{L}_T^2$，则随机过程

$$\int_0^t H_s dB_s := \xi_t$$

为一个$\mathscr{F}_t$ 鞅。

Proof: 适应性和可积性易证，只证鞅性。

考虑简单阶梯过程：

$$H_t = \sum_{i=0}^{n-1} H_{t_i}(\omega)\mathbf{1}_{(t_i,t_{i+1}]}(t) + H_0(\omega)\mathbf{1}_0(t),$$

对任意$u < t \in [0,T]$，我们想要证明

$$\mathbb{E}[\xi_t|\mathscr{F}_u] = \mathbb{E}\left[\int_0^t H_s dB_s|\mathscr{F}_u\right] = \xi_u = \int_0^u H_s dB_s$$

对于划分

$$0 = t_0 < t_1 < \cdots < t_i < \cdots$$

我们把u, t加入原来的划分

$$0 = t_0 < \cdots < t_i < u \le t_{i+1} < \cdots < t_j < t \le t_{j+1} \cdots.$$

记新的划分为

$$0 = t'_0 < t'_1 < \cdots < t'_p < \cdots < t'_q < \cdots,$$

其中$u=t'_p$, $t=t'_q$，且令 $H_{t'_p}=H_{t_i}$，$H_{t'_q}=H_{t_j}$，那么

$$\begin{aligned}
\mathbb{E}[\int_0^t H_s \mathrm{d}B_s|\mathscr{F}_u] &= \sum_{k=0}^{q-1}\mathbb{E}[H_{t'_k}(B_{t'_{k+1}}-B_{t'_k})|\mathscr{F}_{t'_p}] \\
&= \sum_{k=0}^{p-1} H_{t'_k}(B_{t'_{k+1}}-B_{t'_k}) + \sum_{k=p}^{q-1}\mathbb{E}[H_{t'_k}(B_{t'_{k+1}}-B_{t'_k})|\mathscr{F}_{t'_p}] \\
&= \sum_{k=0}^{p-1} H_{t'_k}(B_{t'_{k+1}}-B_{t'_k}) + \sum_{k=p}^{q-1}\mathbb{E}[\mathbb{E}[H_{t'_k}(B_{t'_{k+1}}-B_{t'_k})|\mathscr{F}_{t'_k}]|\mathscr{F}_{t'_p}] \\
&= \sum_{k=0}^{p-1} H_{t'_k}(B_{t'_{k+1}}-B_{t'_k}) + \sum_{k=p}^{q-1}\mathbb{E}[\mathbb{E}H_{t'_k}\mathbb{E}(B_{t'_{k+1}}-B_{t'_k})|\mathscr{F}_{t'_p}] \\
&= \sum_{k=0}^{p-1} H_{t'_k}(B_{t'_{k+1}}-B_{t'_k}) \\
&= \int_0^u H_s \mathrm{d}B_s.
\end{aligned}$$

接下来我们把对简单过程的结果推广到$\mathscr{L}_T^2$。对于$H\in\mathscr{L}_T^2$，存在简单阶梯过程序列$H^{(n)}\in\mathscr{L}_0$逼近H，那么对于任意的$A\in\mathscr{F}_u$

$$\begin{aligned}
\left|\mathbb{E}\left[\int_0^t H\mathrm{d}B\mathbf{1}_A\right]-\mathbb{E}\left[\int_0^t H^{(n)}\mathrm{d}B\mathbf{1}_A\right]\right| &\le \left(\mathbb{E}\left[\int_0^t (H-H^{(n)})\mathrm{d}B\right]^2\right)^{\frac{1}{2}}\mathrm{P}(A)^{\frac{1}{2}}. \\
&= \left(\mathbb{E}\int_0^t (H-H^{(n)})^2\mathrm{d}u\right)^{\frac{1}{2}}\mathrm{P}(A)^{\frac{1}{2}} \overset{n\to\infty}{\longrightarrow} 0.
\end{aligned}$$

同理，

$$\left|\mathbb{E}(\int_0^u H\mathrm{d}B\mathbf{1}_A)-\mathbb{E}(\int_0^u H^{(n)}\mathrm{d}B\mathbf{1}_A)\right| \overset{n\to\infty}{\longrightarrow} 0.$$

注意到对于简单阶梯过程：

$$\mathbb{E}\left[\int_0^t H^{(n)}\mathrm{d}B\mathbf{1}_A\right]=\mathbb{E}\left[\int_0^u H^{(n)}\mathrm{d}B\mathbf{1}_A\right]$$

从而有

$$\mathbb{E}\left[\int_0^t H\mathrm{d}B\mathbf{1}_A\right]=\mathbb{E}\left[\int_0^u H\mathrm{d}B\mathbf{1}_A\right]$$

那么由条件期望的定义，

$$\mathbb{E}[\xi_t|\mathscr{F}_u]=\mathbb{E}\left[\int_0^t H_s dB_s|\mathscr{F}_u\right]=\xi_u=\int_0^u H_s dB_s$$

■

Proposition 10.3 令 $H \in \mathscr{L}_T^2$，则随机过程

$$\eta_t \triangleq \left(\int_0^t H_s dB_s\right)^2 - \int_0^t H_s^2 ds$$

为一个$\mathscr{F}_t$ 鞅。

Proof: 只证鞅性：对任意$u < t \in [0, T]$

$$\mathbb{E}[(\int_0^t H_s \mathrm{d}B_s)^2 - \int_0^t H_s^2 \mathrm{d}s | \mathscr{F}_u] = \left(\int_0^u H_s \mathrm{d}B_s\right)^2 - \int_0^u H_s^2 \mathrm{d}u.$$

注意到

$$\begin{aligned}
LHS &= \mathbb{E}[(\int_0^u H_s \mathrm{d}B_s + \int_u^t H_s \mathrm{d}B_s)^2 - \int_0^u H_s^2 \mathrm{d}s - \int_u^t H_s^2 \mathrm{d}s | \mathscr{F}_u] \\
&= \left(\int_0^u H_s \mathrm{d}B_s\right)^2 - \int_0^u H_s^2 \mathrm{d}s + 2\mathbb{E}[\int_0^u H_s \mathrm{d}B_s \int_u^t H_s \mathrm{d}B_s | \mathscr{F}_u] \\
&+ \mathbb{E}[(\int_u^t H_s \mathrm{d}B_s)^2 - \int_u^t H_s^2 \mathrm{d}s | \mathscr{F}_u]
\end{aligned}$$

我们下面只需要证明：

$$\mathbb{E}[\int_0^u H_s \mathrm{d}B_s \int_u^t H_s \mathrm{d}B_s | \mathscr{F}_u] = 0 \tag{1}$$

$$\mathbb{E}[(\int_u^t H_s \mathrm{d}B_s)^2 - \int_u^t H_s^2 \mathrm{d}s | \mathscr{F}_u] = 0 \tag{2}$$

先证(1)式，

$$\begin{aligned}
LHS &= (\int_0^u H_s \mathrm{d}B_s)\mathbb{E}[\int_u^t H_s \mathrm{d}B_s | \mathscr{F}_u] \\
&= (\int_0^u H_s \mathrm{d}B_s)\mathbb{E}[\int_0^t H_s \mathrm{d}B_s - \int_0^u H_s \mathrm{d}B_s | \mathscr{F}_u] \\
&= 0 \qquad \text{(Itô积分是鞅)}
\end{aligned}$$

再证(2)式，证明过程和证明Itô等距是一样的，不过时间区间换到$[u, t]$，考虑简单阶梯过程

$$H_t = \sum_{i=0}^{n-1} H_{t_i}(\omega) \mathbf{1}_{(t_i, t_{i+1}]}(t) + H_0(\omega) \mathbf{1}_0(t)$$

把u, t加入原来的划分

$$0 = t_0 < \cdots < t_i < u \le t_{i+1} < \cdots < t_j < t \le t_{j+1} \cdots.$$

记新的划分为

$$0 = t'_0 < t'_1 < \cdots < t'_p < \cdots < t'_q < \cdots,$$

其中$u = t'_p$, $t = t'_q$，那么

$$\begin{aligned}
&\mathbb{E}[(\int_u^t H_s \mathrm{d}B_s)^2|\mathscr{F}_u] = \mathbb{E}[\left(\sum_{k=p}^{q-1} H_{t'_k}(B_{t'_{k+1}} - B_{t'_k})\right)^2 |\mathscr{F}_{t'_p}] \\
=&\sum_{k=p}^{q-1} \mathbb{E}[(H_{t'_k}(B_{t'_{k+1}} - B_{t'_k}))^2|\mathscr{F}_{t'_p}] + 2\sum_{p\le k<j\le q-1} \mathbb{E}[H_{t'_k}H_{t'_j}(B_{t'_{k+1}} - B_{t'_k})(B_{t'_{j+1}} - B_{t'_j})|\mathscr{F}_{t'_p}] \\
=&\sum_{k=p}^{q-1} \mathbb{E}[\mathbb{E}[\left(H_{t'_k}(B_{t'_{k+1}} - B_{t'_k})\right)^2|\mathscr{F}_{t'_k}]|\mathscr{F}_{t'_p}] \\
&+ 2\sum_{p\le k<j\le q-1} \mathbb{E}[\mathbb{E}[H_{t'_k}H_{t'_j}(B_{t'_{k+1}} - B_{t'_k})(B_{t'_{j+1}} - B_{t'_j})|\mathscr{F}_{t'_j}]|\mathscr{F}_{t'_p}] \\
=&\sum_{k=p}^{q-1} \mathbb{E}[H^2_{t'_k}(t'_{k+1} - t'_k)|\mathscr{F}_{t'_p}] + 2\sum_{p\le k<j\le q-1} \mathbb{E}[H_{t'_k}H_{t'_j}(B_{t'_{k+1}} - B_{t'_k})\mathbb{E}[(B_{t'_{j+1}} - B_{t'_j})|\mathscr{F}_{t'_j}]|\mathscr{F}_{t'_p}] \\
=&\mathbb{E}[\sum_{k=p}^{q-1} H^2_{t'_k}(t'_{k+1} - t'_k)|\mathscr{F}_{t'_p}] \\
=&\mathbb{E}[\int_u^t H_s^2 \mathrm{d}s|\mathscr{F}_u].
\end{aligned}$$

所以(1)、(2)式成立，故 η 是鞅。 ■

注. 本节课我们说明了所有Itô积分都是鞅，有趣的是，它的逆命题也是成立的，对于带流概率空间$(\Omega, \mathscr{F}, \mathscr{F}_t, P)$上的鞅，都能被表示为Itô积分，这正是著名的鞅表示定理，在金融数学、随机控制中有诸多应用。鞅表示定理的证明需要用到Itô公式，这是我们下节课的内容。